

An Evolutionary Approach to Coalition Formation

Paraskevas V. Lekeas*

June 16, 2014

Abstract

In Cooperative Games with Externalities when the members of a set $S \subset N$ of agents wish to deviate they need to calculate their worth. This worth depends on what the non-members (outsiders) $N \setminus S$ will do, which in turn depends on which coalition structure the outsiders will form. Since this coalition formation problem is NP-hard, various approaches have been adopted. In this paper using an evolutionary game theoretic approach we provide a set of equations that can help agents in S reason about the coalition structures the outsiders may form in terms of minimum distances on an $n - s$ dimensional space, where $n = |N|$, $s = |S|$.

1 Introduction

Coalition Formation in Cooperative Games with Externalities constantly attract the interest of economists. In these games due to cognitive constraints, the members of a coalition cannot accurately predict the coalitional actions of the non-members. This is because deducing the coalitional actions in a game with many players is computationally cumbersome. In [1] it is shown that for an n -player game the number of different coalition structures is $O(n^n)$ and $\omega(n^{\frac{n}{2}})$. Hence, computing the coalition structure that the outsiders form is a particularly difficult task (at least, for games with a large number of players). As a matter of fact, the problem of finding the coalition structure that maximizes the sum of all players' payoffs is NP-hard [1]. Even finding sub-optimal solutions requires the search of an exponential number of cases (e.g. [2, 3]).

In this work we propose a different approach for the coalition formation problem, an Evolutionary approach, stimulated by the following idea (borrowing from [4]):

Classical game theory deals with a rational individual, or 'player', who is engaged in a given interaction or 'game' with other players

* Author is with SimplyHeuristics, Chicago, IL 60631, USA

and has to decide between different options, or ‘strategies’, in order to maximise a ‘payoff’ which depends on the strategies of the co-players (who, in turn, attempt to maximise their payoff). In contrast, evolutionary game theory deals with entire populations of players, all programmed to use some strategy (or type of behaviour). Strategies with high payoff will spread within the population (this can be achieved by learning, by copying or inheriting strategies, or even by infection). The payoffs depend on the actions of the coplayers and hence on the frequencies of the strategies within the population. Since these frequencies change according to the payoffs, this yields a feedback loop. The dynamics of this feedback loop is the object of evolutionary game theory.

In the rest of the paper and in section 2 we present our model, in section 3 we prove the results and in section 4 we conclude.

2 The model

Let $N = \{1, 2, \dots, n\}$ be a number of agents cooperating in a market and let $S \subset N$ be a non-empty set of agents wishing not to cooperate with the rest. Let $N \setminus S$ denote the outsiders. Agents in S are interested in knowing what the outsiders $N \setminus S$ will do upon deviation of S . Let $n = |N|$ and $s = |S|$. Assume that every agent is interchangeably the same as every other agent within the outsiders (and within the whole set of agents in general). Each agent $a \in N \setminus S$ has a number of choices to make as to which other agents in $N \setminus S - \{a\}$ each agent is going to cooperate with in order for the final coalition structure to form. Since the $n - s$ outsiders can be partitioned into disjoint subsets in B_{n-s} ways, B_{n-s} being the $(n - s)^{th}$ Bell number [5], each agent has B_{n-s} choices to follow which reduce (upon symmetry) to $n - s$ different choices since each agent has $n - s$ choices to belong in a coalition of size $1, \dots, n - s$. The following example explains the case for $|N \setminus S| = 3$ outsiders.

Example 2.1 Let $N \setminus S = \{a, b, c\}$ be the set of the outsiders. Since there exist $B_3 = 5$ disjoint partitions (coalition structures) of $N \setminus S$, the choices s_i , $i = 1, \dots, 5$ for agent a are:

- s_1 : a stays a singleton when every other agent stays a singleton $(\{a\}, \{b\}, \{c\})$.
- s_2 : a stays a singleton when every other agent stays together $(\{a\}, \{b, c\})$.
- s_3 : a stays with all the rest $(\{a, b, c\})$.
- s_4 : a stays with b when c stays a singleton $(\{a, b\}, \{c\})$.
- s_5 : a stays with c when b stays a singleton $(\{a, c\}, \{b\})$.

If we now group these choices with respect to the size of the coalition a is a member of, we will end up with 3 different groups of choices, (s_1, s_2) , s_3 , (s_4, s_5) . This happens since a can belong to either a singleton, $(\{a\})$, a couple $(\{a, b\}$ or $\{a, c\})$ or a triple $(\{a, b, c\})$. Observe here that we do not group the coalition

structures into similar ones, i.e. group 1=($\{a, b, c\}$), group 2=($\{\{a, b\}, \{c\}\}$, $\{\{a, c\}, \{b\}\}$, $\{\{b, c\}, \{a\}\}$) and group 3=($\{\{a\}, \{b\}, \{c\}\}$) because we are interested in the decision of agent a with respect to what coalitional size it belongs to. \square

Now agents in S do the following in order to estimate what coalition structure the outsiders will form. They take a very big collection \mathcal{C} of multiple copies of $N \setminus S$ agents and force them to play the following game: Agents in this infinite population interact in groups of $N \setminus S$ forming coalition structures from the set of B_{n-s} possible ones. Then S observes the dynamics of the population and is interested in the following quantity (called the replicator equation)

$$\frac{dx_i}{dt} = x_i(v_{s_i}(a) - \tilde{v}) \quad (1)$$

where x_i is the frequency of choice s_i of agent $a \in N \setminus S$, $v_{s_i}(a)$ is the expected worth of a under choice s_i and \tilde{v} the average worth of the population¹. Assume that each frequency x_i is a differentiable function of time t . As evolution theory suggests (e.g. [6]) the above game has at least one Nash equilibrium which happens when $v_{s_i}(a) - \tilde{v} = 0$. Let us apply the above to our example of $N \setminus S = \{a, b, c\}$.

Example 2.2 *Since in the population we have $B_3 = 5$ different coalition structures and also since we have symmetry (i.e. $v(\{a\}) = v(\{b\}) = v(\{c\})$ and $v(\{a, b\}) = v(\{a, c\}) = v(\{b, c\})$) the average equals to:*

$$\tilde{v} = \frac{1}{3 \cdot B_3} (6v(\{a\}) + 3v(\{a, b\}) + v(\{a, b, c\})) \quad (2)$$

The replicator equation (1) has at least one Nash equilibrium which happens when $v_{s_i}(a) - \tilde{v} = 0$ or for our example when

$$v_{s_i}(\{a\}) = \frac{1}{3 \cdot B_3} (6v(\{a\}) + 3v(\{a, b\}) + v(\{a, b, c\})) \quad (3)$$

which gives a constraint for the outsiders that S should take into account.

As mentioned earlier there exist five choices for agent a but due to symmetry we have that:

$$\begin{aligned} v_{s_1}(a) &= v_{s_2}(a) = v(\{a\}) \\ v_{s_3}(a) &= \frac{v(\{a, b, c\})}{3} \\ v_{s_4}(a) &= v_{s_5}(a) = \frac{v(\{a, b\})}{2} \end{aligned} \quad (4)$$

¹In general with $v(\cdot)$ (with or without subscript) we denote the worth of the argument, the argument being a coalition or a coalition structure.

Using (4) in (3) we have:

$$\begin{aligned} 9v(\{a\}) - 3v(\{a, b\}) - v(\{a, b, c\}) &= 0 \\ -6v(\{a\}) - 3v(\{a, b\}) + 4v(\{a, b, c\}) &= 0 \\ -6v(\{a\}) + \frac{9}{2}v(\{a, b\}) - v(\{a, b, c\}) &= 0 \end{aligned} \quad (5)$$

Equations (5) define three planes in 3-dimensional space (one dimension for each size of a possible non-empty subset of $\{a, b, c\}$). Using simple geometric arguments for every point $P(x, y, z)$ in this space we can find the distances to these planes. So if we know the profits of the coalitions $v(\{a\})$, $v(\{a, b\})$, $v(\{a, b, c\})$ (or else if we can calculate the worth function $v(\cdot)$ of the outsiders) then it is easy to calculate to which plane the worth function lies closer and thus argue as to what coalition structure the outsiders will form. For example, for three dimensions the point $P(x, y, z)$ has the following distances from the above three planes:

$$\begin{aligned} d_{\{a\}} &= 0.105|9x - 3y - z| \\ d_{\{a, b, c\}} &= 0.128|6x + 3y - 4z| \\ d_{\{a, b\}} &= 0.132|6x - 4.5y + z| \end{aligned}$$

So for a characteristic function² with $v(\{a\}) = v(\{b\}) = v(\{c\}) = 0$, $v(\{a, b\}) = v(\{a, c\}) = v(\{b, c\}) = 1$, $v(\{a, b, c\}) = 1$, agents in S will compute that

$$\min\{d_{\{a\}} = 0.419, d_{\{a, b, c\}} = 0.128, d_{\{a, b\}} = 0.463\} = 0.128$$

and estimate that the coalition structure of the outsiders will be $(\{a, b, c\})$. \square

Let us now generalize the above for the case of $n - s$ outsiders.

3 Average worth

In order to generalize the method first we have to compute the average (per agent) worth of the outsiders $N \setminus S$. For this we have the following

Proposition 3.1 *Let $v(j)$ denote the worth of a coalition of j outsiders³. The average worth of the outsiders $N \setminus S$ is*

$$\tilde{v} = \frac{1}{(n - s) \cdot B_{n-s}} \sum_{j=1}^{n-s} v(j) \binom{n-s}{j} B_{n-s-j} \quad (6)$$

²Example borrowed from [7].

³we could have used $v(\{1, \dots, j\})$ instead. From now on these two notations are used interchangeably.

Proof: The $N \setminus S$ outsiders can form B_{n-s} coalition structures in total. Say $\tilde{v}_1, \dots, \tilde{v}_{B_{n-s}}$ is an enumeration of the average worths of all these structures. We have that:

$$\tilde{v} = \sum_{j=1}^{B_{n-s}} \frac{\tilde{v}_j}{B_{n-s}} \quad (7)$$

Every term \tilde{v}_j is a sum of at most $n-s$ worths and this has to be divided by the number of the outsiders $n-s$.⁴

So the sum $\sum_{j=1}^{B_{n-s}} \tilde{v}_j$ is built from the terms $\frac{v(1)}{n-s}, \dots, \frac{v(n-s)}{n-s}$, thus we can write

$$\sum_{j=1}^{B_{n-s}} \tilde{v}_j = \frac{1}{n-s} \sum_{k=1}^{n-s} w_k \cdot v(k) \quad (8)$$

where each weight $w_k, k = 1, \dots, n-s$ denotes the multiplicity (the number of appearances) of a coalition of size k among all the coalitions in all the coalition structures.⁵

We now have to find these weights. To do this we can reason in the following inductive way. Let us first find w_1 , i.e. the weight for the singletons. For this we have to collect all the singletons from all the coalition structures. But we have $n-s$ different ways to choose a singleton from the outsiders since $\binom{n-s}{1} = n-s$, and also the rest $n-s-1$ of the outsiders can form B_{n-s-1} different coalition structures. The above two combined give the weight for the singletons $w_1 = \binom{n-s}{1} B_{n-s-1}$. In the same way we can calculate w_2 : Since we can choose in $\binom{n-s}{2}$ ways a two-size coalition and the rest can form B_{n-s-2} coalition structures, $w_2 = \binom{n-s}{2} B_{n-s-2}$, etc.

So using the above reasoning we have that $w_k = \binom{n-s}{k} B_{n-s-k}$ and using this in (8) and (7) we have the proposition. \square

Example 3.2 Let us use proposition 1 to review our motivating example. For this we have that $n-s = 3$ and $B_3 = 5$. The average worth is

$$\begin{aligned} \tilde{v} &= \frac{1}{3 \cdot 5} \sum_{j=1}^3 v(j) \binom{3}{j} B_{3-j} = \frac{1}{3 \cdot 5} \binom{3}{1} B_2 v(1) + \binom{3}{2} B_1 v(2) + \binom{3}{3} B_0 v(3) \\ &= \frac{1}{3 \cdot 5} (3 \cdot 2 \cdot v(1) + 3 \cdot 1 \cdot v(2) + 1 \cdot 1 \cdot v(3)) \end{aligned}$$

⁴For example if a coalition structure has two structures $\{A\}$ and $\{B\}$ then its average worth is $(v(\{A\}) + v(\{B\})) / (n-s)$, and if the coalition structure has only singletons then the average worth would be a sum of $n-s$ worths of singletons divided by $n-s$: $v(\{1\}) + \dots + v(\{1\}) / (n-s)$ etc.

⁵For example $w_{n-s} = 1$ since there is only one coalition of size $n-s$ among all the coalitions in all the coalition structures.

which is exactly the expression we found in the beginning. Also if we count the multiplicities of singletons, 2-size sets etc. mentioned in Example 1, we indeed find the same quantities. \square

We continue with the following

Proposition 3.3 *Each agent $a \in N \setminus S$ has $n - s$ different choices.*

Proof: Trivially each agent can either exist as a singleton or in a pair or in a triple etc. since the sizes of the non-empty subsets of $N \setminus S$ range in $1, \dots, n$. \square

Let us order the $n - s$ different choices of the agents with respect to the size of the coalition agent a belongs to. Let s'_1, \dots, s'_{n-s} be such an ordering or $(s'_j), 1 \leq j \leq n - s$. Then we have the following proposition

Proposition 3.4 *The average worth of an agent when choice s'_j is adopted is $\frac{v(j)}{j}$.*

Proof: Again trivially, being a member of a j sized coalition and splitting profits equally gives agent a on average the worth of the coalition divided by the number of agents in the coalition which is $\frac{v(j)}{j}$. \square

The following Theorem generalizes the method, giving a computational tool to agents in S to reason about the outsiders

Theorem 3.5 *When $n - s$ outsiders are present, agents in S can reason about the behavior of $a \in N \setminus S$ using the following $n - s$ equations*

$$v(k) \left(\frac{1}{k} - \binom{n-s}{k} \frac{B_{n-s-k}}{(n-s)B_{n-s}} \right) + \sum_{j=1, j \neq k}^{n-s} v(j) \binom{n-s}{j} \frac{B_{n-s-j}}{(n-s)B_{n-s}} = 0$$

$$k = 1, \dots, n - s \quad (9)$$

Proof: Using propositions 1 and 3 in the replicator equation and taking the condition for a Nash equilibrium we get the result. \square

Finally, the following Corollary of Theorem 1 concludes

Corollary 3.6 *Given an allocation vector of worths $(p(1), \dots, p(n-s))$ for the $n - s$ outsiders, the following minimum gives a potential picture to agents in S of the coalition structure the outsiders might form.*

$$\min_k \left\{ \frac{p(k) \left(\frac{1}{k} - \binom{n-s}{k} \frac{B_{n-s-k}}{(n-s)B_{n-s}} \right) + \sum_{j=1, j \neq k}^{n-s} p(j) \binom{n-s}{j} \frac{B_{n-s-j}}{(n-s)B_{n-s}}}{\sqrt{\sum_{j=1, j \neq k}^{n-s} \left(\binom{n-s}{j} \frac{B_{n-s-j}}{(n-s)B_{n-s}} \right)^2 + \left(\frac{1}{k} - \binom{n-s}{k} \frac{B_{n-s-k}}{(n-s)B_{n-s}} \right)^2}} \right\}, k = 1, \dots, n - s$$

$$(10)$$

4 Conclusion

In a set N of agents using an evolutionary approach to the coalition formation problem a set of deviant agents $S \subset N$ might face, we forced the outsiders $N \setminus S$ to arbitrarily form coalition structures in an very big population of multiple copies of $N \setminus S$. After computing the average worth of the outsiders we used the replicator equation to reason about the potential coalition a single agent $a \in N \setminus S$ might belong to. This led to a set of $n - s$ equations (hyperplanes). If agents in S know or have an estimation of the worth of the outsiders, then by computing the distances of this worth to the hyperplanes, they can reason about the coalition structure the outsiders might form.

References

- [1] 1. T. Sandholm, K. Larson, M. Andersson, O. Shehory and F. Tohme. *Coalition Structure Generation with Worst Case Guarantees*. Artificial Intelligence 111 (1-2), 209-238, 1999
- [2] 2. K. Larson, T. Sandholm. Anytime coalition structure generation: an average case study Journal of Experimental & Theoretical Artificial Intelligence 12 (1), 2000
- [3] 3. S. She-Xiong, H. Shan-Li, S. Chun-Yi. Coalition structure generation with worst case guarantees based on cardinality structure, Proceedings of the 6th International joint conference on Autonomous Agents and Multiagent systems (AAMAS '07), Honolulu, Hawaii, 197 (1-3), 2007
- [4] 4. J. Hofbauer and K. Sigmund. *Evolutionary Game Dynamics*. Bulletin of the American Mathematical Society, Vol. 40 no. 4, 479-519, 2003
- [5] 5. E. T. Bell. *Exponential Numbers*. American Mathematical Monthly, 41, 411-419, 1934
- [6] 6. R. Cressman. *Evolutionary Dynamics and Extensive Form Games*. Cambridge, Mass., MIT Press, 2003.
- [7] 7. Y. Kannai *The Core and Balancedness*. In Handbook of Game Theory, Volume 1, Editors R. Aumann and S. Hart. Elsevier Science Publishers, 1992